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Heat diffusion at the boundary of stratified media Homogenized temperature field and thermal constriction

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Abstract

Heat diffusion in stratified materials with the layers running parallel to the main heat flux direction is analyzed with special emphasis on the temperature field near the boundaries. In a previous work, a semi-analytical general solution was proposed as an extension of the thermal quadrupole method for heat conduction in heterogeneous media. In this paper, the steady solution is separated into an homogenized transfer in series with a constriction term, and a conductive boundary layer is defined. The same decomposition method is implemented for the semi-infinite transient case, and some simplified models are obtained from asymptotic expansions. For long times, the transient averaged signal is found to be superposed to the steady constriction matrix effect. The main application is to better envision experimental temperature field analysis for thermal non-destructive evaluation methods.

1. Introduction

The thermophysical characterization of heterogeneous media is a quite difficult problem, due to the multiple spatial scales and characteristic times involved in the heat transfer process, as well as the difficulty to describe the microstructure. Two main approaches are commonly used, as the temperature measurements and processing can be implemented either at the macroscopic level or from the local scale. The local methods consist of heating the sample and measuring the thermal response on a microscopic domain smaller than the spatial characteristic lengths of the components, assuming that the investigated domain is homogeneous at the microscopic scale [1,2]. On the other hand, the classical and widespread transient methods designed for homogeneous materials, such as flash [3] and hot wire [4] methods, can apply at the macroscopic scale for heterogeneous media, but the corresponding macroscopic effective properties must be carefully defined, because their existence and definition strongly depend on the validity of the local thermal equilibrium assumptions.

The homogenization [5] and volume averaging methods [6] yield an in-depth phenomenological analysis allowing both to validate the local thermal equilibrium assumption and specify the relationships between the microstructure, the components properties and the corresponding macroscopic parameters. Quintard and Whitaker [7] achieved a quite general volume averaging approach for the analysis of transient diffusion in two-phase systems, where the generalized volume average is defined from a convolution product with a smooth weighting function. Glatzmaier and Ramirez [8] used a two-equation model to interpret measurements obtained on two-phase samples by the hot wire method, deducing the two effective thermal conductivities and the exchange coefficient. Quintard and Whitaker [7], since they intended to validate their method with the experimental data from [8], showed how these results yield an error due to the fact that the coupled fluxed related to the non-diagonal terms of the macroscopic conductivity

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Nomenclature

a	thermal diffusivity
A	cross-section
b	thermal effusivity
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$	generalized quadrupole matrices
e	thickness
k	thermal conductivity
$\mathbf{K}$	thermal conductivity diagonal matrix $x\Delta\mathbf{z}$
L	length
$\mathbf{M}_c$	constriction matrix
$\mathbf{M}_{//}$	matrix relative to heat transfer versus z
N	nodes number
$\mathbf{P}$	eigenvectors matrix
R_c	thermal constriction resistance
R_a^*	analytical constriction resistance for the average medium
s	Laplace variable
T	temperature
$\overline{T}$	average temperature versus z
$\mathbf{T}_x$	temperature vector at x location
$\mathbf{U}$	averaging matrix
$\mathbf{Z}$	generalized thermal impedance

Greek symbols

δ	conductive boundary layer thickness
$\Delta\mathbf{z}$	space step diagonal matrix
$\mathbf{\Phi}$	heat flux vector
Φ	total heat flux in the x-direction
Ψ	Laplace heat flux vector
φ	heat flux density
θ	Laplace temperature
ρc	volumetric heat capacity
Ω	diagonal eigenvalues matrix

Subscripts

a, b	relative to material a or b
1, 2	relative to medium 1 or 2

Superscripts

*	relative to averaged properties
+	relative to a reduced matrix without the zero eigenvalue
—	averaged variable versus z-direction

tensor were neglected. From thermal diffusivity measurements in a periodic two-layer slab, Truong and Zinmeister [9] suggest that an equivalent homogeneous medium approach is not acceptable when the heat flux is parallel to the layers. A two-equation model is generally needed at the macroscopic scale, because local thermal equilibrium is not fulfilled, especially when the thermal properties of the two constituents differ widely, or when fast transient states are observed.

Although the "change of scale" methods provide a consistent formalism for the implementation of one and two-equation models, they experience important difficulties to specify the boundary conditions at the macroscopic scale [10], and no general formulation seems to be available [11]. Sahraoui and Kaviani [12] introduce a variable effective conductivity near the boundaries of the macroscopic domain. Batsale et al. [13] propose to solve the local problem in the vicinity of boundary, and then couple the corresponding solution to the macroscopic model in the bulk body. The spatial domain where the local model applies is envisioned as a conductive boundary layer. This problem is quite important for the thermal characterization methods, which mostly measure the temperature field at the boundary of the sample. However, such boundary layer is difficult to define, since no systematic approach seems to be available to the authors knowledge.

For homogeneous media, the concept of constriction resistance is widely used in order to give a suitable representation of the two or three-dimensional distortion effects in a globally one-dimensional problem. The thermal constriction resistance R_c is commonly defined, in steady state, for a finite slab of total cross-section A_t and thickness L, with adiabatic lateral walls, as

$$\overline{T}_{A_0}(0) - \overline{T}_{A_t}(L) = \left(\frac{L}{kA_t} + R_c\right)\Phi \tag{1}$$

where $\overline{T}_{A_0}(0)$ is the average temperature over a reduced area A_0 crossed by the heat flux Φ and $\overline{T}_{A_t}(L)$ is the average temperature over A_t. On the right side of Eq. (1), the first term represents the one-dimensional thermal resistance, that is the solution obtained if the heat flux applies over the total cross-section A_t, while the constriction term contains the information relative to the deviation from the one-dimensional case.

Many solutions are available for various geometries in order to calculate this constriction resistance for homogeneous materials: for the half space [14,15], conical asperities [16], finite flux tube [17], or sliding solids [18]. Few works are published about the constriction resistance in heterogeneous media. Negus et al. [19] study the case of a layer coated on a semi-infinite material, with the heat flux perpendicular to the layers. Dryden et al. [20] evaluate the effect of cracks on the thermal resistance of fiber composites, and define a constriction resistance factor that accounts for the effects of both inhomogeneity and geometry. Their analytical solution method is used further in Section 4 for validation.

For transient state, Degiovanni [21] and Degiovanni et al. [22] proposed a simplified model based on integral transforms and asymptotic expansions: the one-dimensional short times thermal impedance is in parallel with the steady state constriction resistance. Some extensions of the flash method designed in order to measure the thermal diffusivity of composite materials with oriented reinforcement were numerically studied by Balageas [23].

The main purpose of this paper is to study heat diffusion in stratified materials with the layers running parallel to the main heat flux direction. Special emphasis is laid on the thermal behavior near the boundaries, that is in the domain where local thermal equilibrium is not achieved.

This problem is fundamental for the thermal characterization methods, since a sensor is to be used on the boundary of the heterogeneous sample. The global objective of this work is to obtain some convenient representations of the transfer matrices between the heat flux and temperature fields at the boundary of a longitudinally stratified medium, in order to implement inverse methods for the cartography of thermophysical properties in heterogeneous media.

The constriction effects near the boundary seem to represent an adequate indicator to quantify the deviation from the homogeneous solution. Thus, the thermal constriction resistance concept is extended here to the multilayered media case.

In next section, the guidelines of the semi-analytical quadrupole approach are given, and the solution of the general problem is pointed out. Then, the solution is split out into two components, the constriction matrix is defined, as well as the conductive boundary layer. In Section 4, the method is validated with an analytical solution in the two-layered slab case, the constriction matrix structure is analyzed and various results are presented. Section 5 is devoted to the extension of this approach to transient state in a semi-infinite medium.

2. The semi-analytical quadrupole approach

The basic thermal quadrupole formalism is an efficient method for multidimensional linear heat conduction modelling and calculation, when involved in multilayer systems [24,25]. For transient conduction in an homogeneous material, a linear intrinsic transfer matrix is relating the input and output temperatures and heat fluxes using some convenient integral transforms. The main advantages of the quadrupole formalism is to make the representation of multilayered systems easy—when the heat flux direction is perpendicular to the layers—by multiplying the corresponding quadrupole matrices, and to avoid gridding the whole domain, as the state variables and fluxes are only calculated on the boundaries. This is an important point when a straightforward relationship between some boundary temperature and heat flux is needed, for instance when dealing with experimental data processing and inverse problems. In a previous work, a general extension of this approach was implemented for heterogeneous media with one-dimensional variation of thermal properties [26], and a semi-numerical general solution was proposed for transient heat transfer in finite or semi-infinite media in both axial and radial coordinate systems, based on a semi-gridding approach. The relationships between the input and output [temperature–heat flux] vectors were written in a matrix form, and some functions of matrix were defined. Such cases are very important for applications to the development of thermal nondestructive evaluation methods by infrared thermography [27] or thermoreflectance measurements [2].

A finite volume grid following the z-direction is applied to the conductive two-dimensional steady state problem shown in Fig. 1(a). It yields the representation depicted in Fig. 1(b), where $\mathbf{T}_x$ and $\mathbf{\Phi}_x$ are respectively the temperature and heat flux vectors at location x corresponding to the N nodes. The general resulting vectorial equation is

$$\mathbf{K}^{-1}\mathbf{M}_{//}\mathbf{T}_x - \frac{\mathrm{d}^2\mathbf{T}_x}{\mathrm{d}x^2} = \mathbf{0} \quad (2)$$

where $\mathbf{T}_x = [\, T_1(x) \quad T_2(x) \quad \cdots \quad \cdots \quad T_N(x)\,]^{\mathrm{t}}$

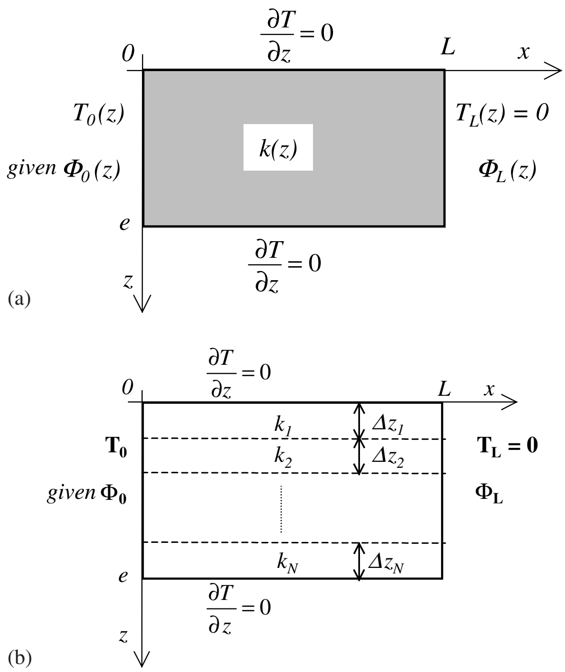

Fig. 1. (a) Two-dimensional steady state problem; (b) Semi-gridding approach.

$$\mathbf{M}_{//} = \begin{bmatrix} H_{1+} & -H_{1+} & 0 & & & & \\ -H_{2-} & H_{2-}+H_{2+} & -H_{2+} & & & & \\ 0 & -H_{3-} & H_{3-}+H_{3+} & -H_{3+} & 0 & & \\ & & \cdots & \cdots & \cdots & & \\ & & & -H_{i-} & H_{i-}+H_{i+} & -H_{i+} & \\ & & & \cdots & \cdots & \cdots & \\ & & & & & -H_{N-} & H_{N-} \end{bmatrix}$$

with $H_{i-} = \left(\frac{\Delta z_{i-1}}{2k_{i-1}} + \frac{\Delta z_i}{2k_i}\right)^{-1}$ and $H_{i+} = \left(\frac{\Delta z_i}{2k_i} + \frac{\Delta z_{i+1}}{2k_{i+1}}\right)^{-1}$ and $\mathbf{K} = \mathbf{diag}([\,k_1\Delta z_1 \quad \cdots \quad k_N\Delta z_N\,])$, where the operator "**diag**" is used in order to build a diagonal matrix from the corresponding vector. The matrix $\mathbf{M}_{//}$ is representative of transverse transfer in the z-direction, while $\mathbf{K}^{-1}$ is the diagonal matrix of lineal thermal resistances versus x.

The heat flux vector is defined as

$$\mathbf{\Phi}_x = -\mathbf{K}\frac{\mathrm{d}\mathbf{T}_x}{\mathrm{d}x} \tag{3}$$

Eq. (2) can be solved directly by the diagonalization:

$$\mathbf{K}^{-1}\mathbf{M}_{//} = \mathbf{P}\Omega\mathbf{P}^{-1} \tag{4}$$

where Ω is the diagonal matrix of eigenvalues, arranged in increasing order. A generalized quadrupole is then written [26] as

$$\begin{bmatrix} \mathbf{T_0} \\ \mathbf{\Phi_0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{T}_\mathrm{L} \\ \mathbf{\Phi}_\mathrm{L} \end{bmatrix} \tag{5}$$

where the generalized quadrupole terms **A**, **B**, **C** and **D** are the $N \times N$ matrices defined as the functions of matrices

$$\mathbf{A} = \mathbf{P}\cosh(\sqrt{\Omega}L)\mathbf{P}^{-1}$$

$$\mathbf{B} = \mathbf{P}\frac{\sinh(\sqrt{\Omega}L)}{\sqrt{\Omega}}(\mathbf{KP})^{-1}$$

$$\mathbf{C} = \mathbf{KP}\sqrt{\Omega}\sinh(\sqrt{\Omega}L)\mathbf{P}^{-1}$$

$$\mathbf{D} = \mathbf{KP}\cosh(\sqrt{\Omega}L)(\mathbf{KP})^{-1}$$

Eq. (5) also applies between any location x in the heterogeneous slab and $x = L$. Assuming the boundary conditions defined in Fig. 1(b), the temperature vector $\mathbf{T}_x$ can be computed as a function of the input heat flux $\mathbf{\Phi_0}$ as

$$\mathbf{T}_x = \mathbf{P}\Big(\sinh(\sqrt{\Omega}(L-x))(\sqrt{\Omega}\cosh(\sqrt{\Omega}L))^{-1}\Big) \times (\mathbf{KP})^{-1}\mathbf{\Phi_0} \tag{6}$$

Eq. (6) is a compact intrinsic relationship, suitable for determining the temperature field as a function of the input heat flux $\mathbf{\Phi_0}$. The temperature field computed with Eq. (6) has been validated in a previous work [26]. It is shown in next section how this equation can be used to build a constriction matrix.

3. Conductive boundary layer and constriction resistance matrix

In this section, the semi-analytical solution given by Eq. (6) is split into (i) the averaged homogeneous part and (ii) the part due to heterogeneous properties, considered as a constriction effect. The conductive boundary layer concept introduced by Batsale [13] is then better envisioned and specified. The two lateral boundary conditions are included in the first and last lines of the matrix $\mathbf{M}_{//}$. Assuming adiabatic lateral boundary conditions, this matrix has a tridiagonal discrete laplacian structure, thus zero is a particular eigenvalue of this matrix. Adiabatic lateral boundary conditions are relevant when studying periodic media. The solution $\mathbf{T}_x^*$ corresponding to the zero eigenvalue is determined by

- $\mathbf{K}^{-1}\mathbf{M}_{//}\mathbf{T}_x^* = \mathbf{0} \Rightarrow T_{x,1}^* = T_{x,2}^* = \cdots = T_{x,N}^*$
- $\frac{\mathrm{d}^2\mathbf{T}_x^*}{\mathrm{d}x^2} = \mathbf{0} \Rightarrow \mathbf{T}_x^*$ is a linear (vector) function of x.

Thus $\mathbf{T}_x^*$ is the one-dimensional linear solution corresponding to the equivalent parallel homogeneous medium such as

$$\mathbf{T}_x^* = \frac{L-x}{k^*e}[\,1 \quad 1 \quad \cdots \quad 1\,]^\mathrm{t}\Phi \quad \text{with}$$

$$k^*e = \sum_1^N k_i\Delta z_i \quad \text{and} \quad \Phi = \sum_1^N \phi_{0,i} \tag{7}$$

where k^* is the equivalent parallel thermal conductivity and Φ is the total input heat flux.

The diagonal matrix of eigenvalues can be written as

$$\Omega = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Omega^+ \end{pmatrix} \tag{8}$$

where Ω^+ is a square matrix of dimension $(N-1)$, containing all eigenvalues but zero, arranged in increasing order. The solution given by Eq. (6) is split out into

$$\mathbf{T}_x = (L-x)\mathbf{P}\begin{pmatrix} 1 & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & \mathbf{0} \end{pmatrix}(\mathbf{KP})^{-1}\boldsymbol{\Phi}_0 + \mathbf{P}\begin{pmatrix} 0 & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & \mathbf{f}(\sqrt{\Omega^+},x) \end{pmatrix}(\mathbf{KP})^{-1}\boldsymbol{\Phi}_0 \tag{9}$$

where

$$\mathbf{f}(\sqrt{\Omega^+},x) = \sinh(\sqrt{\Omega^+}(L-x))(\sqrt{\Omega^+}\cosh(\sqrt{\Omega^+}L))^{-1}.$$

The first term in the right part of Eq. (9) is the homogeneous solution $\mathbf{T}_x^*$. It is important to point out that the zero eigenvalue contribution is equivalent to an averaging operation, as

$$\mathbf{P}\begin{pmatrix} 1 & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & \mathbf{0} \end{pmatrix}\mathbf{P}^{-1} = \frac{1}{k^*e}\begin{pmatrix} k_1\Delta z_1 & k_2\Delta z_2 & \cdots & k_N\Delta z_N \\ k_1\Delta z_1 & k_2\Delta z_2 & \cdots & k_N\Delta z_N \\ \cdots & \cdots & \cdots & \cdots \\ k_1\Delta z_1 & k_2\Delta z_2 & \cdots & k_N\Delta z_N \end{pmatrix} \tag{10}$$

The second term in the right part of Eq. (9) is representative of the two-dimensional transverse constriction effects. The function of matrix $\mathbf{f}$ is the only term depending on the space variable x in Eq. (9). The constriction effects in the medium are obviously negligible when this function is turned to be independent of x.

Practically, when the medium is long enough, the function $\mathbf{f}$ can be approximated by

$$\mathbf{f}\left(\sqrt{\Omega^+},x\right) \approx \exp\left(-\sqrt{\Omega^+}x\right)/\sqrt{\Omega^+} \tag{11a}$$

and this function tends to zero with increasing x, except in a finite layer δ, such as

$$x < \delta = \frac{6}{\sqrt{\Omega^+(1)}} \tag{11b}$$

$$\delta \approx \frac{6e}{\pi} \approx 2e \tag{11c}$$

The approximated value of δ given by Eq. (11c) is deduced from the fact that, for adiabatic lateral boundary conditions, the square root of the eigenvalues is quite close to the eigenvalues $\beta_n = n\pi/e$ of the associated homogeneous medium eigenvalue problem—see [26] for more details. The thickness δ defines a conductive boundary layer, where the constriction effects are effective. Outside of this layer, the temperature field is homogeneous, and is correctly described by $\mathbf{T}_x^*$. It is important to point out that the conductive layer thickness is defined as a maximum value. In some particular cases, when the input perturbation or the thermal contrast between layers is low, the apparent two-dimensional effect could be located quite near the surface.

Applying Eq. (9) at $x = 0$ yields

$$\mathbf{T_0} = \mathbf{T_0^*} + \mathbf{M}_c\boldsymbol{\Phi}_0 \quad \text{with}$$

$$\mathbf{M}_c = \mathbf{P}\begin{pmatrix} 0 & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & \frac{\tanh(\sqrt{\Omega^+}L)}{\sqrt{\Omega^+}} \end{pmatrix}(\mathbf{KP})^{-1} \quad \text{and}$$

$$\mathbf{T_0^*} = \frac{L}{k^*e}[1 \quad 1 \quad \cdots \quad 1]^t\boldsymbol{\Phi} \tag{12a}$$

The decomposition of the temperature vector given by Eq. (12a) for the multilayered medium is quite similar to the previous definition of the constriction resistance for homogeneous materials—see Eq. (1). The first term of the right side of Eq. (12a) is the one-dimensional homogeneous solution corresponding to the zero eigenvalue contribution. The matrix $\mathbf{M}_c$ is a constriction matrix describing the two-dimensional constriction phenomenon that accounts for the effects of both heterogeneity and geometry. The constriction matrix $\mathbf{M_c}$ represents the transverse coupling effects between layers.

For a long shaped medium, an asymptotic expansion can be used for the constriction matrix, such as

$$\lim_{L\to\infty}\mathbf{M}_c = \mathbf{M}_{c,\infty} = \mathbf{P}\begin{pmatrix} 0 & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & (\Omega^+)^{-1/2} \end{pmatrix}(\mathbf{KP})^{-1} \tag{12b}$$

This expression will be use in Section 5 in order to describe transient conduction in a semi-infinite medium. Due to the previous remark about the eigenvalues—see Eq. (11c), the long shaped medium assumption must be understood here as $L \gg e$.

4. Analysis of the constriction resistance matrix

In this section, some results are validated and analyzed for the two-layer slab despicted in Fig. 2(a–c), for various input heat flux boundary conditions.

The temperature field $\mathbf{T}_x$ given by Eq. (6) is plotted in Fig. 3, for the uniform input heat flux shown in Fig. 2(a). The homogeneous part and the constriction term are clearly apparent in Fig. 3. The conductive layer thickness is found to be $\delta = 0.19$ m—Eq. (11b)—quite near to $\delta = 0.20$ m obtained with the approximated value—Eq. (11c). This result is quite consistent with the temperature field as plotted in Fig. 3.

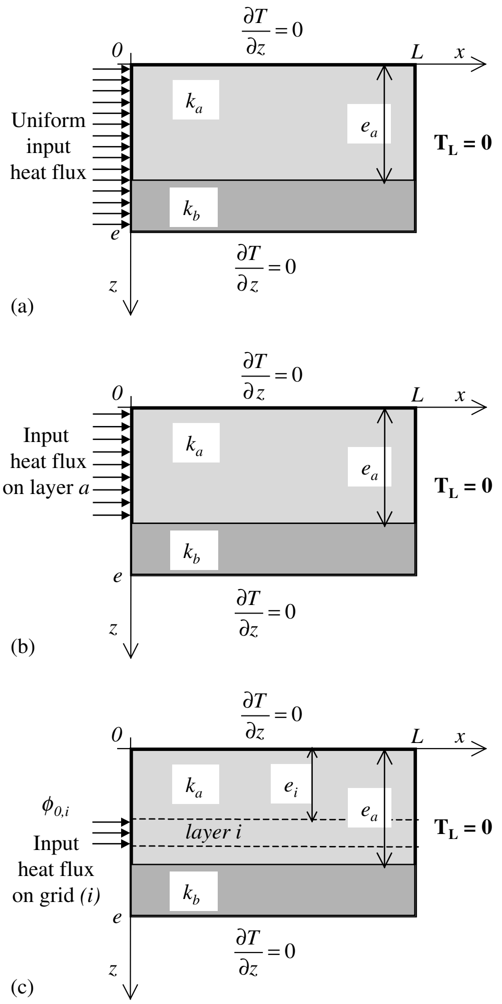

Fig. 2. Two-layer slab subjected to heat flux: (a) Uniform; (b) on layer a; (c) on grid (i) only.

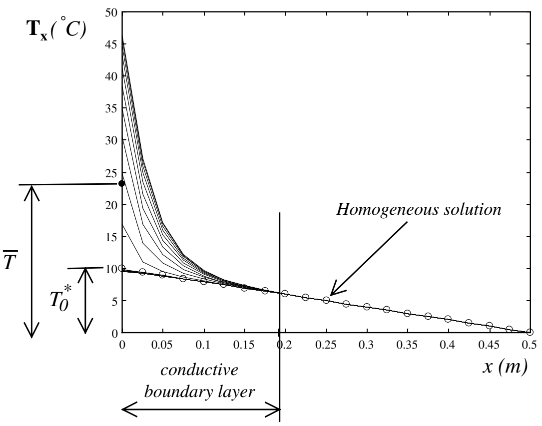

Fig. 3. Temperature field in the two-layer slab with uniform input heat flux: $k_a = 0.01$ W m^{-1} K^{-1}; $k_b = 1$ W m^{-1} K^{-1}; $e = 0.1$ m; $e_a = 0.05$ m; $N = 20$.

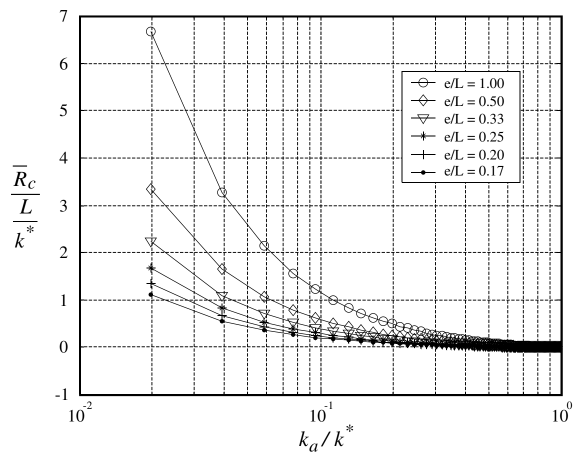

Fig. 4. Constriction resistance: two-layer slab subjected to uniform input heat flux.

When the input heat flux is uniform and the space step Δz is constant, a scalar global constriction resistance can be defined in order to compute a macroscopic one-dimensional relationship between the heat flux density φ and the temperature. This approach is quite useful when an average field is to be used, for instance if $\delta \ll L$. It yields

$$\overline{T} = \frac{L}{k^*}\varphi + \overline{R}_c\varphi \quad \text{with} \quad \overline{R}_c = \frac{1}{e}[1 \quad 1 \quad \cdots \quad 1]\,\mathbf{M}_c[1 \quad 1 \quad \cdots \quad 1]^t \quad \text{and} \quad \overline{T} = \frac{1}{N}[1 \quad 1 \quad \cdots \quad 1]\mathbf{T_0}. \tag{13}$$

The corresponding average temperature $\overline{T}$ is indicated in Fig. 3. The average constriction resistance $\overline{R}_c$ is plotted in dimensionless form in Fig. 4 as a function of the relative thermal conductivity k_a/k^*. The case $k_a/k^* = 1$ is relative to an homogeneous medium, and consequently $\overline{R}_c$ is zero. Decreasing values of e/L correspond to a long shaped medium, and $\overline{R}_c$ tends both to decrease and become independent of L (the curves are closer to each other).

If the input heat flux applies on the whole layer a only (see Fig. 2(b)), the corresponding constriction resistance is obtained by averaging the temperature between $z = 0$ and $z = e_a$. The methodology of solution proposed by Dryden et al. [20] for the radial case is adapted here for the axial case, based on a Fourier cosine transform applied on the x-coordinate. The resulting analytical solution is used in order to validate the present semi-analytical approach. It yields the following analytical solution for the constriction resistance through the layer a:

$$R_{\mathrm{a}} = \frac{L}{k^*e}\frac{k_be_b}{k_ae_a} - \frac{2k_b}{k_aLe_a^2}\sum_{n=1}^{\infty}\left[\frac{\sinh(\alpha_ne_a)(\sinh(\alpha_ne_b)}{\alpha_n^3(k_a\cosh(\alpha_ne_b)\sinh(\alpha_ne_a)+k_b\cosh(\alpha_ne_a)\sinh(\alpha_ne_b))}\right] \quad \text{with } \alpha_n = (2n-1)\pi/2L \tag{14}$$

This analytical solution is compared to the constriction resistance obtained from the constriction matrix (Eq. (12a)), and plotted in Fig. 5 as a function of the relative thermal conductivity k_a/k^* for various location of the a/b interface. The agreement between the analytical solution and the semi-analytical solution is quite good. The relative thermal conductivity k_a/k^* tends to unity when the material tends to be homogeneous. This limiting case yields the constriction resistance due to geometrical effects only. This means that the remaining part of the curves mostly shows the heterogeneity effects—that is transverse transfer due to the variation of the thermal conductivity.

Another situation is helpful to analyze the constriction matrix, when the input heat flux is non-zero only on a given layer (i), as shown in Fig. 2(c). Applying Eq. (12a) yields:

$$T_i = \frac{L}{k^*e}\phi_{0,i} + \mathbf{M}_{\mathrm{c}}(i,i)\phi_{0,i} \tag{15a}$$

Since T_i is the average temperature of layer i (with respect to the control volume), the diagonal term $\mathbf{M}_{\mathrm{c}}(i,i)$ represents the corresponding constriction resistance for this layer. In order to compare this term with the constriction resistance in the equivalent homogeneous medium k^*, with the same geometrical characteristics, the analytical solution of the problem shown in Fig. 2(c) is implemented from a Fourier cosine transform applied on the z-coordinate, and yields the following analytical constriction resistance:

$$R_{\mathrm{a}}^* = \frac{2}{k^*e\Delta z^2}\sum_{n=1}^{\infty}\tanh(\beta_nL)\frac{[\sin(\beta_n(e_i+\Delta z))-\sin(\beta_ne_i)]^2}{\beta_n^3} \tag{15b}$$

where $\beta_n = n\pi/e$ are the eigenvalues of the associated boundary value problem.

Eq. (15b) is found to be consistent with the results given by Laraqi [18], if the velocity is set to zero, and $L \gg e$. The integral transform applies on z and not on x as for Dryden's approach.

The local relative constriction effect in the two-layer slab, as defined in Fig. 2(c), is plotted in Fig. 6 as a function of the dimensionless thermal conductivity contrast and the location of the interface between the layers. As expected, the dimensionless constriction resistance is found to tend to unity when the thermal conductivity contrast tend to one, that is when the medium is homogeneous. When the layer i where the input heat flux applies is located in the more conductive phase ($e_a < e_i$ or $e_a/e < 0.50$, corresponding to the line with points in Fig. 6), heat transfer is almost one-dimensional in the conductive layer, the insulating layer is mostly unperturbated, and the relative constriction resistance tend to 1: $\mathbf{M}_{\mathrm{c}}(i,i) \approx R_{\mathrm{a}}^*$. When the layer i is located in the insulating phase ($e_a/e \geqslant 0.50$), the constriction effect is strongly increased, except if the medium is quasi-homogeneous (when k_a/k^* tends to 1).

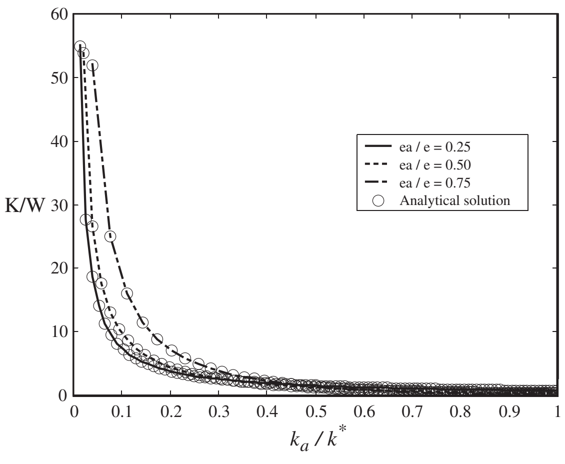

Fig. 5. Constriction resistance: two-layer slab subjected to input heat flux on layer a.

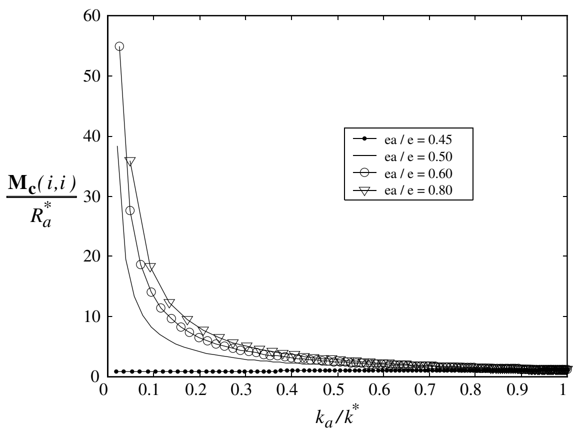

Fig. 6. Local constriction resistance: two-layer slab subjected to heat flux on grid (i) only; $k_b = 1\ \mathrm{W\,m^{-1}\,K^{-1}}$; $e = 0.1$ m; $e_i/e = 0.5$; $L = 0.5$ m; $N = 20$.

In next section, it is shown how this approach can be extended to the transient case in order to study the transient constriction effects at the boundary of a semi-infinite longitudinally stratified medium.

5. Transient conduction at the boundary of a semi-infinite stratified medium

Studying transient heat conduction at the boundary of a semi-infinite longitudinally stratified medium is relevant for various measurements technics, such as non-destructive testing by infrared camera, cartography of thermophysical properties in heterogeneous media or photoreflectance images treatment, where some convenient transfer functions between the superficial temperature field and heat flux are required, in order to implement suitable inverse methods.

For transient state, a Laplace transform relative to time is applied to the vector $\mathbf{T}_x(t)$ as

$$\theta_x(s) = \int_0^\infty \exp(-st)\mathbf{T}_x(t)\,\mathrm{d}t \tag{16}$$

Eq. (2) is turned into

$$(\mathbf{K}^{-1}\mathbf{M}_{//} + \mathbf{a}^{-1}s)\theta_x - \frac{\mathrm{d}^2\theta_x}{\mathrm{d}x^2} = \mathbf{0} \tag{17a}$$

where the vector θ_x is the Laplace transform of $\mathbf{T}_x$ and $\mathbf{a}$ is the diagonal matrix of the thermal diffusivities: $\mathbf{a} = \mathbf{diag}([\,a_1 \quad \cdots \quad a_N\,])$. The matrix in Eq. (17a) is diagonalized as

$$(\mathbf{K}^{-1}\mathbf{M}_{//} + \mathbf{a}^{-1}s) = \mathbf{P}_\mathrm{L}\Omega_\mathrm{L}\mathbf{P}_\mathrm{L}^{-1} \tag{17b}$$

and the transfer matrix between the Laplace input temperature and heat flux vectors is found for the semi-infinite medium as a generalized thermal impedance [26]:

$$\theta_\mathbf{0} = \mathbf{Z}\Psi_0 \tag{18a}$$

where the vector $\Psi_\mathbf{0}$ is the Laplace transform of heat flux vector $\Phi_\mathbf{0}$, and the transfer matrix $\mathbf{Z}$ is defined as a product between a function of matrix and the thermal conductivity matrix such as

$$\mathbf{Z} = (\mathbf{K}^{-1}\mathbf{M}_{//} + \mathbf{a}^{-1}s)^{-1/2}\mathbf{K}^{-1} = \mathbf{P}_\mathrm{L}(\Omega_\mathrm{L})^{-1/2}\mathbf{P}_\mathrm{L}^{-1}\mathbf{K}^{-1} \tag{18b}$$

Previous considerations about the smallest eigenvalue separation in steady state can be extended to this transient case through an examination of the function of matrix defined by Eq. (18b), in order to find some simplified representations of the transfer matrix. Some asymptotic expansions for short times and long times can be considered.

5.1. Short times asymptotic expansion

For short times, the square matrix on the left side of Eq. (17a) can be approximated by the single product $\mathbf{a}^{-1}s$, the matrix $\mathbf{K}^{-1}\mathbf{M}_{//}$ vanishes, and the corresponding heat transfer is one-dimensional for each layer. The expression "short times" should be understood as "low Fourier numbers relative to the corresponding layers $a_i t/\Delta z_i^2$". The diagonalization is no more necessary, and the solution given by Eq. (18) is approximated as

$$\theta_{\mathbf{0},\text{short time}} = \frac{\sqrt{\mathbf{a}}}{\sqrt{s}}\mathbf{K}^{-1}\Psi_\mathbf{0} = \frac{\mathbf{b}^{-1}\Delta\mathbf{z}^{-1}}{\sqrt{s}}\Psi_\mathbf{0} \tag{19}$$

where $\mathbf{b}$ is the diagonal matrix of thermal effusivities $(i = 1 - N : b_i = \sqrt{k_i(\rho c)_i})$ and $\Delta\mathbf{z}$ is the space step diagonal matrix. If a step heat flux is applied, then the short times temperature response is a linear function of the square root of time, and the slop of each layer depends on the local thermal effusivity—see Fig. 9. This result is consistent with the analytical results about transient constriction in homogeneous media as given by Degiovanni [21].

5.2. Long times asymptotic expansion: simplified model

The asymptotic expansion for long times corresponds to the limit when s tend to zero. In that case, the averaging characteristics of the zero eigenvalue and the previous steady state decomposition, such as Eq. (9), would suggest that s could be neglected in the diagonal matrix, except for the term corresponding to the zero eigenvalue, such as

$$\lim_{s\to 0}\mathbf{Z} \approx \sqrt{a^*}s^{-1/2}\mathbf{P}\begin{pmatrix} 1 & [0 \quad \cdots \quad 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & \mathbf{0} \end{pmatrix}(\mathbf{KP})^{-1} + \mathbf{P}\begin{pmatrix} 0 & [0 \quad \cdots \quad 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & (\Omega^+)^{-1/2} \end{pmatrix}(\mathbf{KP})^{-1} \tag{20a}$$

where a^* is the average thermal diffusivity calculated with the average thermal conductivity and volumetric heat capacity.

The form of Eq. (20a) means that for long times, a transient averaged part is superposed to the steady state constriction matrix effect. This approximation would be valid if

$$\frac{s}{a^*} \ll \Omega^+(1) \iff t \gg \tau_{\mathrm{lt}} = \frac{1}{a^*\Omega^+(1)} \approx \frac{e^2}{a^*\pi^2} \tag{20b}$$

that is when the penetration depth of heat in the homogenized medium is greater that $e/3$. Eq. (20b) also means that the penetration depth has the same order of magnitude that the steady conductive boundary layer defined in Eq. (11b). In that case, Eq. (18a) is turned into an equivalent transient form of the previous steady state decomposition given by Eq. (12a), such as

$$\theta_{\mathbf{0},\text{long time}} \approx \theta_{\mathbf{0}}^* + \mathbf{M}_{c,\infty} \boldsymbol{\Psi}_{\mathbf{0}} \tag{21}$$

where $\mathbf{M}_{c,\infty}$ is the steady constriction matrix defined in Eq. (12b),

$\theta_{\mathbf{0}}^* = \dfrac{1}{b^* e\sqrt{s}} \mathbf{U} \boldsymbol{\Psi}_{\mathbf{0}}$ represents the one dimensional averaged contribution,

$\mathbf{U} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ is an averaging matrix,

$b^* = \sqrt{k^*(\rho c)^*}$ is the average thermal effusivity of the medium,

$(\rho c)^* e = \sum_1^N \rho c_i \Delta z_i$ is the average volumetric heat capacity.

The function of matrix used in Eq. (18b) is neither additive nor separable, and no exact separated solution can be deduced from Eq. (18a). The simplified model proposed by Eq. (21) is only an approximation of the exact model for long times. Moreover, when the thermal diffusivity matrix is uniform (each layer has the same thermal diffusivity a), then the exact solution for long times matches exactly the simplified model, thus

$$\mathbf{Z}_{a=\text{cste}} = \left(\mathbf{K}^{-1}\mathbf{M}_{//} + \frac{s}{a}\mathbf{I}\right)^{-1/2}\mathbf{K}^{-1} = \mathbf{P} \times \left(\Omega + \frac{s}{a}\mathbf{I}\right)^{-1/2}\mathbf{P}^{-1}\mathbf{K}^{-1} \tag{22a}$$

$$\lim_{s\to 0} \mathbf{Z}_{a=\text{cste}} = \mathbf{P}\begin{pmatrix} \sqrt{a}s^{-1/2} & [0 \ \cdots \ 0] \\ \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} & (\Omega^+)^{-1/2} \end{pmatrix}(\mathbf{KP})^{-1} \tag{22b}$$

Eq. (22b) yields exactly the simplified model given by Eq. (21). In the general case where the thermal diffusivity is not uniform in the layers, Eq. (18b) tends asymptotically to Eq. (21) when the thermal diffusivity tends to be uniform.

For the two nodes model ($N = 2$), the complete analytical solution of Eq. (18b) is available, since the eigenvectors and eigenvalues matrices corresponding to Eq. (4) can be calculated analitically. The resulting generalized impedance has the following form

$$\mathbf{Z_2} = \left(\mathbf{P_2}\begin{pmatrix} s/a^* & \omega_{12}s \\ \omega_{21}s & \omega_{22}s + \Omega \end{pmatrix}\mathbf{P_2}^{-1}\right)^{-1/2}\begin{pmatrix} k_1 e_1 & 0 \\ 0 & k_2 e_2 \end{pmatrix}^{-1} \tag{23a}$$

where the non-diagonal terms of the inner matrix are found to be zero when $a_1 = a_2$, $\mathbf{P}_2$ is the eigenvectors matrix and Ω the non-zero eigenvalue corresponding to Eq. (4).

It is apparent in Eq. (23a) that the inner matrix has almost a diagonal structure when s tend to zero. Moreover, the characteristic equation of this inner matrix shows that its eigenvalues, with a first order approximation in the Laplace variable s, are s/a^* and Ω, thus

$$\lim_{s\to 0} \mathbf{Z_2} \approx \mathbf{P_2}\begin{pmatrix} \sqrt{a^*}s^{-1/2} & 0 \\ 0 & 1/\sqrt{\Omega} \end{pmatrix}(\mathbf{KP_2})^{-1} \tag{23b}$$

and the validity of the resulting approximation with the simplified model corresponding to Eq. (21) is demonstrated for the two nodes case. When the thermal diffusivity profile differ widely, a longer time must be reached for the simplified model to be valid.

Anyway, the behavior of the temperature field for long times can be computed and investigated systematically from the semi-analytical model given by Eqs. (18), and then be compared with the simplified model corresponding to Eq. (21). A numerical Laplace transform inversion is performed, using a Gaver–Stehfest algorithm [28]. For all the cases investigated, the approximation of Eq. (21) is found to be acceptable, as shown on Fig. 7, where the long time temperature profiles obtained from both models for the two-layer slab of Fig. 2(a) are plotted as a function of z for various thermal diffusivity ratio, for an instant corresponding to ten

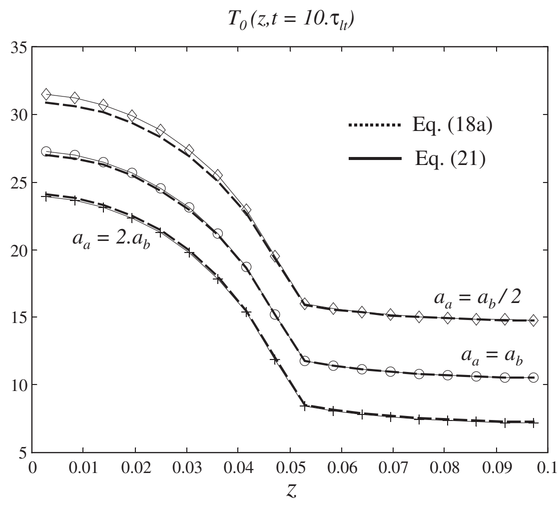

Fig. 7. Long times asymptotic expansion. comparison with the simplified model: $k_a = 0.1\ \mathrm{W\,m^{-1}\,K^{-1}}$; $k_b = 1\ \mathrm{W\,m^{-1}\,K^{-1}}$; $\rho c_a = 10^4\ \mathrm{J\,m^{-3}\,K^{-1}}$; $e_a/e = 0.5$.

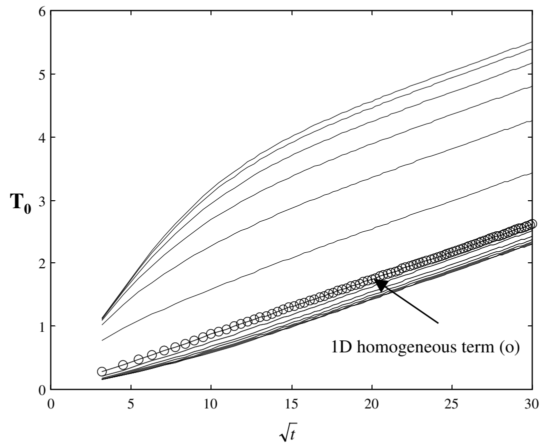

Fig. 8. Long times asymptotic expansion: transient average term and steady constriction effect. $k_a = 0.1$ W m^{-1} K^{-1}; $k_b = 1$ W m^{-1} K^{-1}; $\rho c_a = 10^4$ J m^{-3} K^{-1}; $\rho c_b = 5 \times 10^4$ J m^{-3} K^{-1}; $\tau_{lt} = 55$ s.

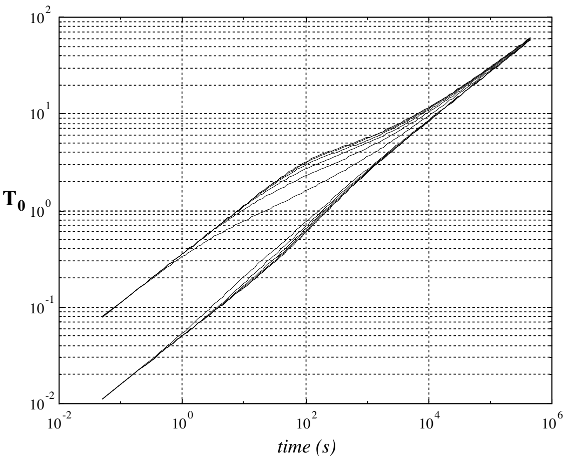

Fig. 9. Input temperature field as a function of the square root of time. Characteristic times: short times = 3 s; $\tau_{lt} = 55$ s.

times the characteristic time given by Eq. (20b), where $\tau_{lt} = 140$ s.

The long times input temperature vector $\mathbf{T_0}$ is computed from the semi-analytical model corresponding to Eqs. (18) and plotted on Fig. 8 as a function of the square root of time, for a two-layer semi-infinite slab and a uniform input step heat flux, such as Fig. 2(a). For homogeneous materials, this case is illustrative of the hot film method, designed in order to measure the thermal effusivity of homogeneous semi-infinite materials [29]. For long times, the curves become parallel to the average solution. As expected from Eq. (21), the general solution is composed of two contributions: the transient average term is added to the steady constriction matrix effect. The transient homogeneous term is a linear function of $\sqrt{t}$, and the slope is found to be the inverse of the average thermal effusivity. The results are consistent with the calculated characteristic time τ_{lt}.

The same case is presented in Fig. 9, for a larger time domain. It is apparent from this logarithmic plot that both short and long times correspond to some linear functions of the square root of time, since the slop is 1/2. For short times, the two curves correspond to the thermal effusivities of both layers, as predicted by Eq. (19). The characteristic time for short times expansion, given by the Fourier numbers relative to the corresponding layers is in this case about three seconds. For long times, the curves exhibit the dependance on $\sqrt{t}$ of the one-dimensional average medium.

6. Conclusions

Heat conduction at the interface of a stratified heterogeneous medium, with insulated lateral walls, can be split out into an equivalent average one-dimensional resistance in series with a constriction matrix term. A conductive boundary layer can then be defined, where two-dimensional effects occur, and the thickness of this layer can be evaluated. For transient heat conduction in a semi-infinite stratified medium, the well known results about the one-dimensional behavior for short times are extended to the long times case through an asymptotic expansion study. A transient averaged signal corresponding to the equivalent homogeneous medium is found to be superposed to the steady constriction matrix contribution. The governing parameter of such approximation is thermal diffusivity. When the thermal diffusivity profile differ widely, a longer time must be reached for this approximation to be valid.

These results could be quite useful for the implementation of inverse methods for thermal properties measurement technics such as thermoreflectance, flash or hot probe methods, when applied to heterogeneous media with one-dimensional varying properties. This new approach could be implemented in a radial coordinate system and in a three-dimensional form, in order to be applied to fibrous materials. The semi-analytical solutions proposed in this paper contribute to better envision the problem of including the boundary conditions effect in the homogenization methods.

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